

Calculus III: Coursework 3
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The deadline is 4PM on Tuesday Oct 16th. Hand-in is to the green box on the ground floor of the Maths building.

1. Sketch the curves whose parametric equations are [23]

(a) $\mathbf{r} = (2 \cos \pi t)\mathbf{i} + (\sin \pi t)\mathbf{j}$

(b) $\mathbf{r} = (\sin \pi t)\mathbf{i}$

(c) $\mathbf{r} = t\mathbf{i} + |t|\mathbf{k}$

$(-\infty \leq t \leq \infty)$, and write down the derivatives $d\mathbf{r}/dt$ and $d^2\mathbf{r}/dt^2$ where they are defined.

Answer

(a) This is an ellipse with semi-major axes length 2 and 1 along the x and y axes respectively, centred at the origin.

$$\begin{aligned}\frac{d\mathbf{r}}{dt} &= (-2\pi \sin \pi t)\mathbf{i} + (\pi \cos \pi t)\mathbf{j} \\ \frac{d^2\mathbf{r}}{dt^2} &= (-2\pi^2 \cos \pi t)\mathbf{i} + (-\pi^2 \sin \pi t)\mathbf{j}\end{aligned}$$

(Motion in each of x and y is simple harmonic.)

(b) The interval $[-1, 1]$ in x .

$$\begin{aligned}\frac{d\mathbf{r}}{dt} &= \pi \cos \pi t \mathbf{i} \\ \frac{d^2\mathbf{r}}{dt^2} &= -\pi^2 \sin \pi t \mathbf{i}\end{aligned}$$

(Simple harmonic motion)

(c) The lines $z = -x$ for $x = t \leq 0$ and $z = x$ for $x = t > 0$ in the (x, z) plane.

In $t < 0$, $\frac{d\mathbf{r}}{dt} = \mathbf{i} - \mathbf{k}$ and in $t > 0$, $\frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{k}$. In both regions the second derivative is $\mathbf{0}$ (Note: the zero vector, not the scalar 0). The curve is not differentiable at $t = 0$.

Marks:

For curves 4,2,3 respectively. For evidence of understanding (in at least one of the parts) of how to form the derivatives 4, for correct answers for them 4,2,4 respectively. Lose 1 in the last part unless it is clear curve is not differentiable at $t = 0$.

2. Find the relation of dv/dt to $d\mathbf{v}/dt$ for any non-zero vector $\mathbf{v}$. Hence show that [10]

$$dv/dt = 0 \Rightarrow d\mathbf{v}/dt \perp \mathbf{v}.$$

Answer

Since $v^2 = \mathbf{v} \cdot \mathbf{v}$ we have

$$v \frac{dv}{dt} = \mathbf{v} \cdot \frac{d\mathbf{v}}{dt}.$$

If the LHS is zero we must have

$$\frac{d\mathbf{v}}{dt} \cdot \mathbf{v} = 0$$

which is the result.

Marks: Using $v^2 = \mathbf{v} \cdot \mathbf{v}$ 3, differentiating correctly 3 (note that for a parameter u ,

$$\frac{d(\mathbf{F} \cdot \mathbf{G})}{du} = \mathbf{F} \cdot \frac{d\mathbf{G}}{du} + \frac{d\mathbf{F}}{du} \cdot \mathbf{G}.$$

was given in lectures), knowing that $\perp$ means zero dot product and using it 4.

3. Calculate $\nabla \cdot \mathbf{F}$ (i.e. $\text{div } \mathbf{F}$) for each of the following vector fields. [13]

(a) $\mathbf{F} = 3xyz^2\mathbf{i} + y^2 \sin z\mathbf{j} + xe^{2z}\mathbf{k}$

(b) $\mathbf{F} = 5z\mathbf{i} + 3y\mathbf{j} + 4z\mathbf{k}$

Answer

(a) $\nabla \cdot \mathbf{F} = 3yz^2 + 2y \sin z + 2xe^{2z}$

(b) $\nabla \cdot \mathbf{F} = 0 + 3 + 4 = 7$

Marks: 5 for a correct method for div , each part 4 for accuracy.

4. Calculate $\nabla \times \mathbf{F}$ (i.e. $\text{curl } \mathbf{F}$) for each of the following vector fields. [19]

(a) $\mathbf{F} = (2xy, x^2 + z^2, yz)$

(b) $\mathbf{F} = (e^x \sin y, e^x \cos y, \cos z)$

Answer

(a)

$$\begin{aligned} \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy & x^2 + z^2 & yz \end{vmatrix} \\ &= \mathbf{i}(z - 2z) + \mathbf{j}(0 - 0) + \mathbf{k}(2x - 2x) = -z\mathbf{i}. \end{aligned}$$

(b)

$$\begin{aligned} \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^x \sin y & e^x \cos y & \cos z \end{vmatrix} \\ &= \mathbf{i}(0 - 0) + \mathbf{j}(0 - 0) + \mathbf{k}(e^x \cos y - e^x \cos y) = \mathbf{0}. \end{aligned}$$

(Comment: in fact this is a manifestation of $\nabla \times \nabla V = \mathbf{0}$ for any V , since this $\mathbf{F}$ is $\nabla(e^x \sin y - \sin z)$.)

Marks: 5 for a correct method for curl, each part 7 for accuracy (these need more calculation than divergences).

5. Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = 3y^2\mathbf{i} - 2xz\mathbf{j} + 4xz\mathbf{k}$, over each of the following curves from $(0, 0, 0)$ to $(1, 1, 1)$: [25]

(a) $C: \mathbf{r} = t\mathbf{i} + t\mathbf{j} + t\mathbf{k} \quad 0 \leq t \leq 1$

(b) $C: \mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + t^4\mathbf{k} \quad 0 \leq t \leq 1$

(c) C consisting of the line segment from $(0, 0, 0)$ to $(1, 1, 0)$ followed by the line segment from $(1, 1, 0)$ to $(1, 1, 1)$.

Answer

(a) On this curve $d\mathbf{r} = (\mathbf{i} + \mathbf{j} + \mathbf{k})dt$ and $\mathbf{F} = 3t^2\mathbf{i} - 2t\mathbf{j} + 4t^2\mathbf{k}$ so we have

$$\int_0^1 (3t^2 - 2t + 4t^2)dt = [7t^3/3 - t^2]_0^1 = 4/3.$$

(b) On this curve $d\mathbf{r} = (\mathbf{i} + 2t\mathbf{j} + 4t^3\mathbf{k})dt$ and $\mathbf{F} = 3t^4\mathbf{i} - 2t\mathbf{j} + 4t^5\mathbf{k}$ so we have

$$\int_0^1 (3t^4 - 4t^2 + 16t^8)dt = [16t^9/9 + 3t^5/5 - 4t^3/3]_0^1 = 47/45.$$

(c) First part is $\mathbf{r} = t\mathbf{i} + t\mathbf{j}$ and the second part is $\mathbf{r} = \mathbf{i} + \mathbf{j} + t\mathbf{k}$, each with $0 \leq t \leq 1$. On first part $\mathbf{F} = 3t^2\mathbf{i} - 2t\mathbf{j}$ and on the second part $\mathbf{F} = 3\mathbf{i} - 2\mathbf{j} + 4t\mathbf{k}$. So from the first part we get

$$\int_0^1 (3t^2 - 2t)dt = [t^3 - t^2]_0^1 = 0$$

while the second part gives

$$\int_0^1 4t dt = [2t^2]_0^1 = 2$$

which is also the total.

Marks: For evidence of a correct method for forming $d\mathbf{r}$ 5, for a correct method of evaluating $\mathbf{F}$ on a curve and forming the integral 5, for accuracy 4,4,7 per part. Last part gets more because of the need to find the curve parametrization etc.

6. If $\mathbf{F} = -\alpha(x\mathbf{i} + y\mathbf{j})$ (where α is a constant) along the path [10]

$$\mathbf{r} = (\cos \pi t)\mathbf{i} + (\sin \pi t)\mathbf{j} + t\mathbf{k} \quad (0 \leq t \leq 5),$$

find $\int \mathbf{F} \cdot d\mathbf{r}$. [A physical basis for this would be a magnetic field exerting the force $\mathbf{F}$ on a charged particle, making the particle move along a spiral path. The integral would be the work done by the force]

Answer

On the curve $d\mathbf{r} = [(-\pi \sin \pi t)\mathbf{i} + (\pi \cos \pi t)\mathbf{j} + \mathbf{k}]dt$ and $\mathbf{F} = -\alpha((\cos \pi t)\mathbf{i} + (\sin \pi t)\mathbf{j})$ so the integral is

$$\int_0^5 \pi\alpha(\sin \pi t \cos \pi t - \cos \pi t \sin \pi t)dt = 0.$$

Marks: For correct method and forming integral 5, for accuracy 5.

(Note: the force acts to keep the x, y motion circular but is always perpendicular to the velocity. Thus the speed of the body, and its energy, stay constant – no work is done.)